Faculty Working Papers

THE MARKET MODEL: POTENTIAL FOR ERROR

Dennis J. Collins, Assistant Professor
Department of Accountancy
James C. McKeown, Professor, Department of Accountancy

#606

College of Commerce and Business Administration
University of Illinois at Urbana-Champaign

Faculty Working Papers

ASYMPTOTIC EXPANSIONS OF THE DISTRIBUTIONS OF THE TEST STATISTICS FOR OVERIDENTIFYING RESTRICTIONS IN A SYSTEM OF SIMULTANEOUS EQUATIONS

Kimio Morimune, Associate Professor, Department of Economics
Naoto Kunitomo, Stanford University
Yoshihiko Tsukuda, Yamagata University

#616

College of Commerce and Business Administration
University of Illinois at Urbana-Champaign

FACULTY WORKING PAPERS

College of Commerce and Business Administration

University of Illinois at Urbana-Champaign

October 10, 1979

ASYMPTOTIC EXPANSIONS OF THE DISTRIBUTIONS OF THE TEST STATISTICS FOR OVERIDENTIFYING RESTRICTIONS IN A SYSTEM OF SIMULTANEOUS EQUATIONS

Kimio Morimune, Associate Professor, Department of Economics
Naoto Kunitomo, Stanford University
Yoshihiko Tsukuda, Yamagata University

#616

Summary:

We derive the asymptotic expansions of the distributions of test statistics for overidentifying restrictions for a linear structural equation. We analyze two test statistics: one is associated with the limited information maximum likelihood estimator, and the other is associated with the fixed k-class estimator. We also apply two kinds of asymptotic expansions: one is the large sample asymptotics, and the other is the small-disturbance asymptotics. Therefore we obtain four resulting expansions of the distributions.

Acknowledgment:

Professor Morimune's work is supported in part by NSF Grant SOC76-22232; and Professor Kunitomo is partially supported by NSF Grant SOC77-14949 at the Institute of Mathematical Studies in the Social Sciences at Stanford University.

1. Introduction

Anderson and Rubin (1949) proposed the likelihood-ratio statistic for testing the overidentifiability conditions on a structural equation in a system of simultaneous equations. The likelihood-ratio statistic is found to be equivalent to the smallest root of the determinantal equation in the theory of the Limited Information Maximum Likelihood (LIML) estimator. Anderson and Rubin (1950) found that $T\lambda_{LIML}$ (see 2.8) is asymptotically distributed as χ^2 with L degree of freedom. Here L is the degree of overidentification of a structural equation, and T is the sample size.

Basmann (1960) proposed that $((T - K)/L)\lambda_{LIML}$ be compared to a Fisher's variance ratio distribution with L and T - K degrees of freedom (K is the number of exogeneous variables in the system.) Additionally he proposed an alternative test statistic (λ_{TSLS} here) based on the Two-Stage Least-Squares (TSLS) estimator of structural parameters. In other words, the numerator of $(\lambda_{TSLS} + 1)$ is the residual sum of squares of a structural equation associated with the TSLS estimation. The test of significance for λ_{TSLS} is the same as that for λ_{LIML}. His proposals were supported by a Monte Carlo experiment on a set of parameters and exogenous variables.

Later, Basmann (1966) derived the exact distribution of λ_{TSLS} for a specific case. Richardson (1968) derived the exact distribution of λ_{TSLS} for the case of two endogenous variables. He proved that the exact moments of λ_{TSLS} approach to those of the F distribution as the noncentrality parameter (concentration parameter) goes to infinity. McDonald (1972) derived the exact distribution of λ_{LIML} for the case of two endogenous variables. He proved that Richardson's results on the expansion of exact moments also hold for λ_{LIML}. Rhodes (1978) derived the exact distribution of λ_{LIML} using zonal polynomials. The structural equation in his study includes many endogenous variables.

Kadane (1970) demonstrated that the F distribution conjectured by Basmann is the first order term of asymptotic expansions of λ_{LIML} and λ_k (λ with any fixed k-class estimator) in the variance of the disturbance about zero. The meaning of his asymptotic series called the small-disturbance asymptotics was clarified by Anderson (1977). That is to say, the noncentrality parameter grows indefinitely large while the sample size stays fixed in the small-disturbance asymptotics, and the sample size increases together with the noncentrality parameter in the large sample asymptotic theory.

There were some disputes on the interpretations implied by the null-hypothesis of testing overidentifiability. Ambiguity on this matter was clarified by Kadane and Anderson (1977).

We derive asymptotic expansions of the distributions of λ_{LIML} and λ_k for the two parameter sequences. The first term of approximate distributions of $(T - K)\lambda_{LIML}$ and $(T - K)\lambda_{TSLS}$ is the χ^2 distribution with L degree of freedom in the large sample asymptotics. The first term of approximate distribution of $((T - K)/L)\lambda_{LIML}$ and $((T - K)/L)\lambda_k$ is the F distribution with L and (T - K) degrees of freedom in the small-disturbance asymptotics. We can see the difference of distributions of λ_{LIML} and λ_k comparing the higher order terms.

Model and test statistics are defined in Section 2. The resulting approximate distributions of λ_{LIML} and λ_k are stated in Section 3. In

Section 4 empirical small sample distributions of λ_{LIML}, λ_{TSLS} and λ_{OLS} are presented for particular models. We also discuss about deviations of the approximate distributions from small sample distributions. The proof is given in Appendix.

2. Model and Test Statistics

Let a single structural equation be

$$y_1 = Y_2\beta + Z_1\gamma + u, \tag{2.1}$$

where y_1 and Y_2 are $T \times 1$ and $T \times G_1$ matrices, respectively, of observations on the endogenous variables, Z_1 is a $T \times K_1$ matrix of observations on the K_1 exogenous variables, β and γ are column vectors with G_1 and K_1 unknown parameters, and u is a column vector of T disturbances. We assume that (2.1) is the first equation in a simultaneous system of $G_1 + 1$ linear stochastic equations relating $G_1 + 1$ endogenous variables and $K(K = K_1 + K_2)$ exogenous variables. The reduced form of $Y = (y_1\ Y_2)$ is defined as

$$\begin{aligned} Y &= Z\Pi + V \\ &= (Z_1\ Z_2)(\pi_1\ \Pi_2) + (v_1\ V_2) \end{aligned} \tag{2.2}$$

where Z is a $T \times K$ matrix of exogeneous variables (full rank), $\pi_1' = (\pi_{11}'\ \pi_{21}')$ and $\Pi_2' = (\Pi_{12}'\ \Pi_{22}')$ are, respectively, $1 \times (K_1 + K_2)$ and $G_1 \times (K_1 + K_2)$ matrices of the reduced form coefficients, and $(v_1\ V_2)$ is a $T \times (1 + G_1)$ matrix of disturbances. We make the following assumptions about the model:

ASSUMPTION 1. The rows of $\underset{\sim}{V}$ are independently normally distributed, each row having mean $\underset{\sim}{0}$ and (nonsingular) covariance matrix

$$\underset{\sim}{\Omega} = \begin{pmatrix} \omega_{11} & \underset{\sim}{\omega}_{12} \\ \underset{\sim}{\omega}_{21} & \underset{\sim}{\Omega}_{22} \end{pmatrix} \tag{2.3}$$

ASSUMPTION 2. The matrix $(\underset{\sim}{\pi}_{21}\ \underset{\sim}{\Pi}_{22})$ is of rank G_1 and $\underset{\sim}{\Pi}_{22}$ is also of rank G_1.

In order to relate (2.1) and (2.2) postmultiply (2.2) by $(1, -\underset{\sim}{\beta}')'$, then $\underset{\sim}{u} = \underset{\sim}{v}_1 - \underset{\sim}{V}_2\underset{\sim}{\beta}$, $\underset{\sim}{\gamma} = \underset{\sim}{\pi}_{11} - \underset{\sim}{\Pi}_{12}\beta$ and

$$\underset{\sim}{\pi}_{21} = \underset{\sim}{\Pi}_{22}\underset{\sim}{\beta}. \tag{2.4}$$

This is the null hypothesis of testing overidentifiability (Kadane and Anderson, 1977). The components of $\underset{\sim}{u}$ are independently normally distributed with mean 0 and variance

$$\sigma^2 = \omega_{11} - 2\underset{\sim}{\beta}'\underset{\sim}{\omega}_{21} + \underset{\sim}{\beta}'\underset{\sim}{\Omega}_{22}\underset{\sim}{\beta}. \tag{2.5}$$

Let

$$\underset{\sim}{C} = \underset{\sim}{Y}'(\underset{\sim}{I}_T - \underset{\sim}{Z}(\underset{\sim}{Z}'\underset{\sim}{Z})^{-1}\underset{\sim}{Z}')\underset{\sim}{Y} \tag{2.6}$$

and

$$\underset{\sim}{G} = \underset{\sim}{Y}'(\underset{\sim}{Z}(\underset{\sim}{Z}'\underset{\sim}{Z})^{-1}\underset{\sim}{Z}' - \underset{\sim}{Z}_1(\underset{\sim}{Z}_1'\underset{\sim}{Z}_1)^{-1}\underset{\sim}{Z}_1')\underset{\sim}{Y}, \tag{2.7}$$

then the test statistic for overidentifiability is

$$\lambda = \frac{(1 - \hat{\beta}')G(1 - \hat{\beta}')'}{(1 - \hat{\beta}')C(1 - \hat{\beta}')'} \tag{2.8}$$

where $\hat{\beta}$ is the k-class estimator of β. We denote λ_{LIML} as λ with LIML estimator of β, and λ_k with any fixed k-class estimator including λ_{TSLS} and λ_{OLS}. We use the following notation in theorems:

$$L = K_2 - G_1, \tag{2.9}$$

$$f^* = -\frac{\nu\sqrt{\omega_{11.2}}}{\sigma} \Theta_{22}^{-1/2}\Omega_{22}^{1/2}\alpha \tag{2.10}$$

$$\mu^2 = 1/\mathrm{tr}\{(I + \alpha\alpha')^{-1}\Omega_{22}^{1/2}(\Pi_{22}'A_{22.1}\Pi_{22})^{-1}\Omega_{22}^{1/2}\} \tag{2.11}$$

where

$$A_{22.1} = Z_2'Z_2 - Z_2'Z_1(Z_1'Z_1)^{-1}Z_1'Z_2, \tag{2.12}$$

$$\omega_{11.2} = \omega_{11} - \omega_{12}\Omega_{22}^{-1}\omega_{21}, \tag{2.13}$$

$$\alpha = \frac{1}{\sqrt{\omega_{11.2}}} \Omega_{22}^{-1/2}(\Omega_{22}\beta - \omega_{21}). \tag{2.14}$$

We additionally assume in the large sample asymptotics as:

<u>ASSUMPTION 3:</u> $$\Theta_{22} = \Pi_{22}'A_{22.1}\Pi_{22}/T + O(T^{-2}). \tag{2.15}$$

This assumption is natural since $A_{22.1}$ is a moment matrix of exogenous variables. We also define

$$\nu^2 = \mu^2/T. \tag{2.16}$$

The definition (2.16) exists by Assumption 3. In the small disturbance asymptotics the definition of $\underset{\sim}{\Theta}_{22}$ should be conceived without the higher order term in (2.15). Instead, $\underset{\sim}{f}^*$ is assumed to be constant to terms of order μ^{-2}. We refer to L as the degree of over-identification and to μ^2 as the noncentrality parameter.

3. Distributions of Overidentifiability Test Statistics

The magnitude of the structural variance decreases in the small disturbance asymptotics. It leads that the noncentrality parameter (2.11) grows while the sample size is fixed in the small-disturbance analysis since the orders of magnitude of $\underset{\sim}{\Omega}$ and σ^2 are the same (Anderson 1977; Kunitomo et. al. 1979). The following theorem follows when the disturbances are small.

Theorem 1: When μ^2 increases while the sample size stays fixed,

$$\Pr\{\frac{T - K}{L} \lambda_k \leq \xi\} = F_{L,T-K}(\xi) + \frac{1}{\mu^2}\{\xi - \frac{L(L + T - K)}{T - K + L\xi} \underset{\sim}{f}^{*\prime}\underset{\sim}{f}^{*}\xi^2$$

$$- (1 - k)[\frac{T - K}{L}(1 - k) + 2\xi][1 + \frac{(T - K)(L + T - K)}{T - K + L\xi} \underset{\sim}{f}^{*\prime}\underset{\sim}{f}^{*}]\}f_{L,T-K}(\xi) \quad (3.1)$$

and

$$\Pr\{\frac{T - K}{L} \lambda_{LIML} \leq \xi\} = F_{L,T-K}(\xi) + \frac{\xi}{\mu^2}\{1 + \frac{L}{T - K} \xi\}f_{L,T-K}(\xi) \quad (3.2)$$

to terms of order μ^{-2}, and $F_{L,T-K}$ and $f_{L,T-K}$ are F-distribution and density functions with L and T-K degrees of freedom, respectively.

The first term of (3.1) and (3.2) are derived by Kadane (1970). Approximate distributions of the test statistics associated with the TSLS and OLS estimators are (3.1) with k = 1 and 0, respectively. It is interesting to see that neither (3.1) nor (3.2) include terms of order μ^{-1}. It is rather striking to see that the nuisance parameter included in (3.2) is merely μ^2. The term of order μ^{-2} of (3.2) is positive always. Then we expect that a critical value imposed by a F distribution is greater than that implied by an exact distribution if μ^2 is sufficiently large.

Turning to the large sample asymptotics, the next theorem follows.

Theorem 2: When μ^2 increases together with the sample size,

$$\Pr\{(T-K)\lambda_{TSLS} \leq \xi\} = G_L(\xi) + \frac{\xi}{\mu^2}\{1 + \frac{\nu^2}{2}(L - 2 - \xi) - \underset{\sim}{f}^{*\prime}\underset{\sim}{f}^{*}\xi\}g_L(\xi) \tag{3.3}$$

and

$$\Pr\{(T-K)\lambda_{LIML} \leq \xi\} = G_L(\xi) + \frac{\xi}{\mu^2}\{1 + \frac{\nu^2}{2}(L - 2 - \xi)\}g_L(\xi) \tag{3.4}$$

to terms of order T^{-1}, and G_L and g_L are the χ^2 distribution and density functions with L degree of freedom.

The first term of (3.4) is given by Anderson and Rubin (1950). Fujikoshi (1977) derived (3.4) with different parametrization in MANOVA. We find again that the second order terms $(T^{-1/2})$ do not appear in expansions.

4. Monte Carlo Studies

It is easy to give numerical evaluations of approximate distributions. However, exact distributions seem hard to evaluate (see Richardson 1968; and McDonald 1972). Then, instead of exact distributions, empirical distributions by Monte Carlo studies are obtained for the purpose of comparisons between small sample and approximate distributions. The empirical small sample distributions calculated for our purpose are

$$\Pr\{\frac{T - K}{K_2 - 1} \lambda \leq \xi\}. \tag{4.1}$$

All the approximations are standardized to be comparable with (4.1).

About ten models are chosen from Anderson, Morimune and Sawa (1979) for experiments, and we show results on four of them. See Anderson, Morimune and Sawa (1979) for the references and discussions on properties of these models. The number of observations is 20,000 in Monte Carlo experiments. The computer time was relatively short since we utilize canonical representations of test statistics. It is easily seen that the parameters to be specified in experiments are T, K, L, f* and μ^2 in virtue of canonical forms.

In the following discussion on the results of experiments "F-approximations" mean the asymptotic expansions in Theorem 1, and "χ^2-approximations" mean the asymptotic expansions in Theorem 2.

As for Table 1, both the F- and χ^2-approximations ((2), (3), (5), and (6)) give much better approximations to the empirical small sample distributions ((1) and (4)) than the simple F and χ^2 distributions ((8) and (9)). The F-distribution (8) is a better approximation than the χ^2-distribution (9) to the distribution of λ_{LIML}, and it may be considered as a good substitute for the distribution of λ_{LIML}; but not for that of λ_{TSLS}. This is because the value of α (it is -f* when $G_1 = 1$) is relatively large. The distribution of λ_{OLS} is extremely deviated from the other distributions, and the F-approximate distribution does not give good approximation, either (not shown in Table 1).

As for Table 2 the F and χ^2-approximations ((2), (3), (5), and (6)) are improved from the simple F and χ^2-distributions ((8) and (9)), respectively. This can be true for the LIML as well as TSLS estimators. It seems, however, that the F-approximations are giving better approximations than the χ^2-approximations. Moreover, it may be adequate to say that the F-distribution (8) is a good substitute for the distribution of λ_{LIML} (1) but not for λ_{TSLS} (4).

As for Table 3 the F-approximations ((2) and (5)) seem to be better than the χ^2-approximations ((3) and (6)) again. Especially, the F-approximation of λ_{TSLS} (5) is quite close to the empirical small sample distribution (4). With respect to this Model the sample F-distribution (8) is a good substitute of the small-sample distribution of λ_{TSLS} (4). The small sample distribution of λ_{LIML} (1) is also close to F, but it deviates from F at higher values of ξ. The distribution of λ_{OLS} (7) gets closer to the other distributions than in the previous two models.

As for Table 4 the χ^2-distribution (8) is deviated from the small sample distributions ((1), (4), and (5)). However, the other distributions ((2), (3), (6), and (7)) give close approximations to the small sample distributions. As far as approximations of λ_{LIML} and λ_{TSLS} are concerned, LIML and TSLS give numerically the same distributions since α is relatively so small compared with μ^2. A different phenomenon from previous models is that the small sample distribution of λ_{OLS} (5) is relatively close to those of λ_{LIML} and λ_{TSLS}. This is also because the value of μ^2 is large compared wilth the value of (T - K).

5. Discussion

The critical values (e.g., at the 95% level) given by χ^2 distributions after standardizations according to (4.1) are smaller than those given by empirical distributions; the latters are again smaller than those given by F distributions for λ_{LIML} as far as our limited number of experiments are concerned. The critical values given by F-approximations stay between empirical and F distributions. Therefore testing overidentifiability using χ^2 distribution is severer than it should be if we do not want to reject a null-hypothesis. Testing overidentifiability using F distribution is more lenient than it is supposed to be. It may also be fair to say that F distributions give more accurate critical values than χ^2 distributions in our experiments. Nevertheless, it may be less accurate to use F or χ^2 distributions for the distribution of λ_{TSLS}.

Takeuchi and Morimune (1979) proposed the extended limited information maximum likelihood (ELI) estimator which is third order efficient. Similarly they proposed the extended two-stage least-squares (ETS) estimator, and it was found that the ETS estimator is less efficient than the ELI estimator. It was also noted that the ELI estimator is asymptotically equivalent to Fuller's modified limited information estimator (1977). The large sample asymptotic distributions of λ (2.8) with the ELI or ETS estimator (denoted as λ_{ELI} and λ_{ETS}) are

$$\Pr\{(T - K)\lambda_{ELI} \leq \xi\} = \Pr\{(T - K)\lambda_{LIML} \leq \xi\} - \frac{h^2}{\mu^2} \underset{\sim}{f}^{*\prime}\underset{\sim}{f}^{*} g_L(\xi), \tag{5.1}$$

and

$$\Pr\{(T - K)\lambda_{ETS} \leq \xi\} = \Pr\{(T - K)\lambda_{LIML} \leq \xi\} - \frac{1}{\mu^2}(L + h - \xi)^2 \underset{\sim}{f}^{*\prime}\underset{\sim}{f}^{*} g_L(\xi) \tag{5.2}$$

to terms of order T^{-1}, respectively, and an h is any real value determining the asymptotic bias of ELI.

Morimune (1978) proposed an estimator which linearly combines the LIML and TSLS estimators such as $\hat{\beta}_{COM} = (L - 1)\hat{\beta}_{LIML}/L + \hat{\beta}_{TSLS}/L$. The distribution of the variance ratio for this estimator is expanded as

$$\Pr\{(T - K)\lambda_{COM} \leq \xi\} = \Pr\{(T - K)\lambda_{LIML} \leq \xi\} - \frac{\xi^2}{\mu^2 L^2} f^{*\prime} f^* g_L(\xi), \tag{5.3}$$

as the sample size increases to terms of order T^{-1}, and

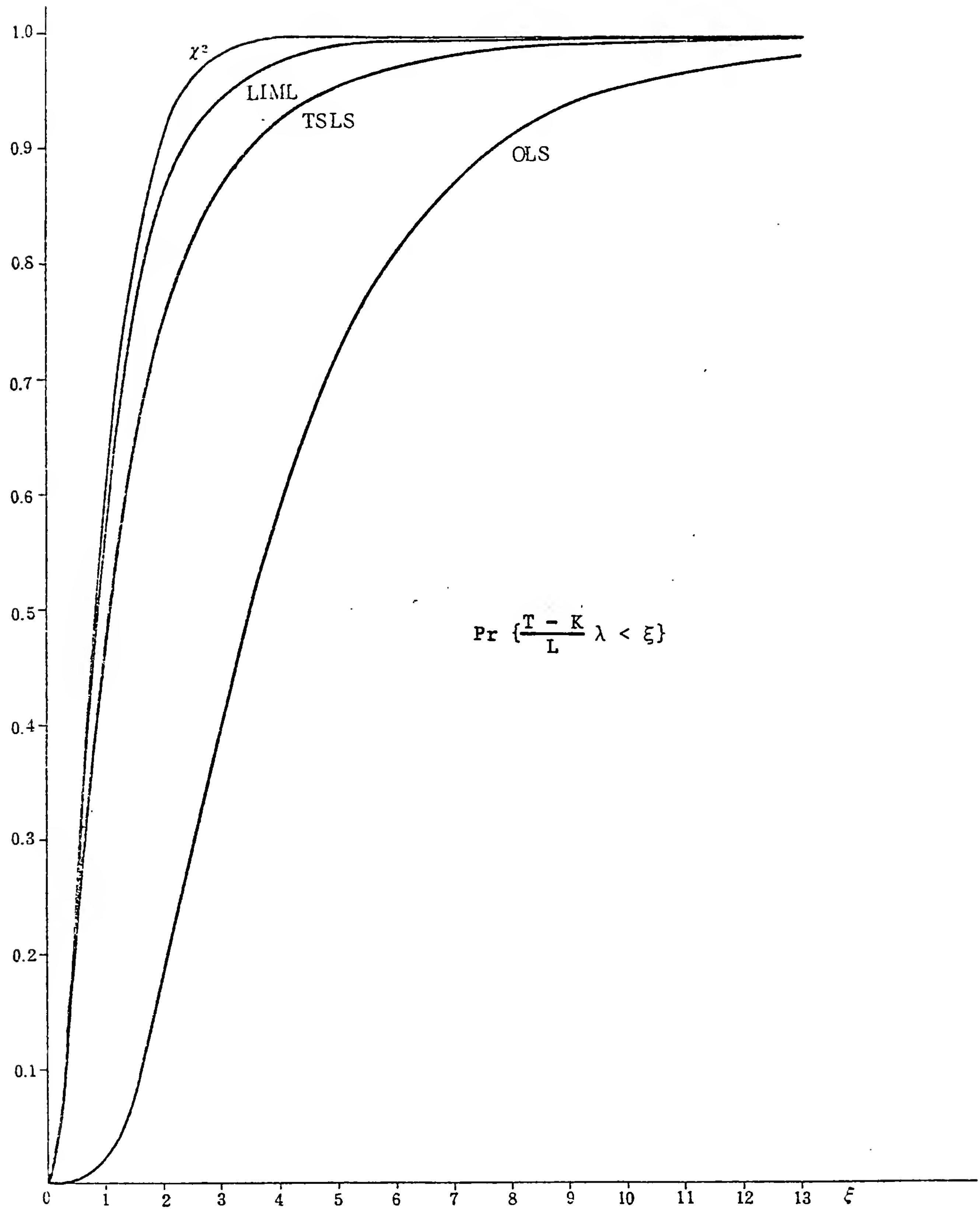
$$\Pr\{\frac{T - K}{L} \lambda_{COM} \leq \xi\} = \Pr\{\frac{T - K}{L} \lambda_{LIML} \leq \xi\}$$

$$- \frac{\xi^2}{\mu^2 L} \{\frac{1}{T - K} + \frac{L + T - K}{T - K + L\xi} f^{*\prime} f^*\} f_{L,T-K}(\xi) \tag{5.4}$$

as μ^2 increases to terms of order μ^{-2}.

Basmann (1960) proceeded a Monte Carlo experiments for a model whose parameters are summarized as $\mu^2 = 10.52$, $f^{*\prime} f^* = 0.42$, $T - K = 10$, and $L = 1$.

Model 2



χ^2 with 13 d.f., and empirical small sample distributions of λ_{LIML}, λ_{TSLS}, and λ_{OLS} are given. The F distribution with 4 and 13 d.f., and the F-approx, and χ^2-approx of λ_{LIML} are indistinguishable from the distribution of λ_{LIML}. F-approx and χ^2-approx of λ_{TSLS} are not distinguishable from the distribution of λ_{TSLS}, either.

Table 1

Model 1 ($T=20$, $K=5$, $K_2=2$, $\alpha=1.985$, $\mu^2=42.3$: Girshick-Haavelmo (3.4))

The percentage points are given for each distribution according to values of ξ on the leftest column

ξ	(1) Empirical (LIML)	(2) F-approx (LIML)	(3) χ^2-approx (LIML)	(4) Empirical (TSLS)	(5) F-approx (TSLS)	(6) χ^2-approx (TSLS)	(7) Empirical (OLS)	(8) F	(9) χ^2
0.5	0.517	0.515	0.515	0.502	0.504	0.504	0.001	0.510	0.521
1.0	0.669	0.673	0.672	0.646	0.651	0.650	0.003	0.667	0.683
2.0	0.829	0.828	0.827	0.786	0.791	0.788	0.020	0.822	0.843
3.0	0.904	0.901	0.900	0.858	0.861	0.857	0.049	0.896	0.912
4.0	0.940	0.940	0.939	0.901	0.902	0.899	0.093	0.936	0.955
5.0	0.965	0.962	0.962	0.927	0.928	0.928	0.150	0.959	0.975
7.0	0.986	0.983	0.984	0.959	0.960	0.963	0.278	0.982	0.992

The values of ξ are given for each distribution according to percentages on the leftest column

%	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
25%	0.10	0.10	0.10	0.11	0.10	0.10	6.61	0.11	0.10
50%	0.46	0.47	0.47	0.50	0.49	0.49	10.69	0.48	0.46
75%	1.40	1.40	1.40	1.66	1.61	1.63	*	1.43	1.32
90%	2.93	2.98	3.00	3.98	3.95	4.04		3.07	2.70
95%	4.30	4.40	4.42	6.32	6.22	6.10		4.53	3.84

* the number of observations is not enough to give accurate figures below

Table 2

Model 2 ($T=21$, $K=8$, $K_2=5$, $\alpha=-2.072$, $\mu^2=74.9$: Klein 1(I))

The percentage points are given for each distribution according to values of ξ on the leftest column

ξ	(1) Empirical (LIML)	(2) F-approx (LIML)	(3) χ^2-approx (LIML)	(4) Empirical (TSLS)	(5) F-approx (TSLS)	(6) χ^2-approx (TSLS)	(7) Empirical (OLS)	(8) F	(9) χ^2
0.5	0.270	0.269	0.269	0.227	0.224	0.227	0.002	0.264	0.264
1.0	0.565	0.566	0.560	0.470	0.456	0.436	0.022	0.558	0.594
2.0	0.854	0.853	0.845	0.744	0.734	0.710	0.176	0.846	0.908
3.0	0.947	0.945	0.950	0.866	0.869	0.888	0.397	0.941	0.983
4.0	0.978	0.977	0.986	0.924	0.933	0.966	0.585	0.975	0.997
5.0	0.990	0.990	0.996	0.955	0.964	0.991	0.720	0.988	1.0

The values of ξ are given for each distribution according to percentages on the leftest column

%	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
25%	0.47	0.47	0.47	0.54	0.55	0.55	2.34	0.48	0.48
50%	0.87	0.87	0.88	1.07	1.12	1.21	3.50	0.88	0.84
75%	1.50	1.50	1.55	2.04	2.09	2.18	5.30	1.53	1.35
90%	2.38	2.38	2.41	3.49	3.40	3.08	7.75	2.43	1.94
95%	3.06	3.09	3.01	4.80	4.46	3.69	9.74	3.18	2.37

Model 3 ($T=20$, $K=7$, $K_2=4$, $\alpha=-0.304$, $\mu^2=21.4$: Klein 3 (3.3.38))

The percentage points are given for each distribution according to values of ξ on the leftest column

ξ	(1) Empirical (LIML)	(2) F-approx (LIML)	(3) χ^2-approx (LIML)	(4) Empirical (TSLS)	(5) F-approx (TSLS)	(6) χ^2-approx (TSLS)	(7) Empirical (OLS)	(8) F	(9) χ^2
0.5	0.328	0.328	0.327	0.323	0.324	0.325	0.184	0.311	0.318
1.0	0.605	0.600	0.594	0.596	0.590	0.588	0.449	0.576	0.608
2.0	0.860	0.856	0.846	0.844	0.844	0.838	0.765	0.836	0.888
3.0	0.947	0.944	0.939	0.933	0.934	0.935	0.897	0.930	0.971
4.0	0.979	0.976	0.977	0.968	0.969	0.975	0.950	0.968	0.993
5.0	0.991	0.989	0.992	0.984	0.984	0.991	0.974	0.984	0.998

The values of ξ are given for each distribution according to percentages on the leftest column

%	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
25%	0.39	0.39	0.39	0.40	0.39	0.39	0.62	0.41	0.40
50%	0.78	0.79	0.79	0.79	0.80	0.80	1.12	0.83	0.79
75%	1.44	1.45	1.49	1.49	1.50	1.52	1.92	1.55	1.37
90%	2.34	2.37	2.46	2.51	2.50	2.53	3.04	2.56	2.08
95%	3.06	3.13	3.20	3.36	3.35	3.28	4.01	3.41	2.60

Table 4

Model 4 ($T=20$, $K=6$, $K_2=3$, $\alpha=-0.0183$, $\mu^2=634.1$: Klein 3 (3.3.46))

The percentage points are given for each distribution according to values of ξ on the leftest column

ξ	(1) Empirical (LIML)	(2) F-approx (LIML & TSLS)	(3) χ^2-approx (LIML & TSLS)	(4) Empirical (TSLS)	(5) Empirical (OLS)	(6) F-approx (OLS)	(7) F	(8) χ^2
0.5	0.379	0.384	0.383	0.378	0.372	0.376	0.383	0.393
1.0	0.611	0.608	0.606	0.611	0.606	0.603	0.607	0.632
2.0	0.836	0.828	0.826	0.837	0.835	0.828	0.828	0.865
3.0	0.921	0.918	0.918	0.921	0.919	0.917	0.918	0.950
4.0	0.960	0.958	0.961	0.959	0.958	0.957	0.958	0.982
5.0	0.979	0.977	0.981	0.979	0.979	0.977	0.977	0.993

The values of ξ are given for each distribution according to percentages on the leftest column

%	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
25%	0.30	0.29	0.29	0.30	0.31	0.31	0.29	0.29
50%	0.73	0.73	0.73	0.73	0.75	0.74	0.73	0.69
75%	1.49	1.53	1.54	1.49	1.50	1.55	1.53	1.39
90%	2.65	2.72	2.73	2.66	2.67	2.74	2.73	2.30
95%	3.68	3.73	3.67	3.68	3.71	3.76	3.74	3.00

Appendix A

We freely use the results, symbols, and random variables defined by Fujikoshi et. al. (1978) and Kunitomo et. al. (1979) for deriving the asymptotic expansions of distributions of variance ratios. The difference in notation in this paper and those in Fujikoshi et. al. are

$$\underset{\sim}{f}^* = \nu \underset{\sim}{f} \tag{A-1}$$

$$\underset{\sim}{F}^* = \nu \underset{\sim}{F} \tag{A-2}$$

where $\underset{\sim}{F} = \underset{\sim}{\Theta}_{22}^{-1/2}\underset{\sim}{\Omega}_{22}^{1/2}(\underset{\sim}{I} + \underset{\sim}{\alpha}\underset{\sim}{\alpha}')^{-1/2}$, and $\underset{\sim}{f}$ and $\underset{\sim}{F}$ are used in Fujikoshi et. al. The new notations are used since it renders simplicity to analyses which include two alternative sequences. The variance ration (2.8) is rewritten for any estimator as

$$\lambda = \frac{(1 - \hat{\underset{\sim}{\beta}}')\underset{\sim}{Q}^{-1}\underset{\sim}{G}^*\underset{\sim}{Q}'^{-1}(1 - \hat{\underset{\sim}{\beta}}')'}{(1 - \hat{\underset{\sim}{\beta}}')\underset{\sim}{Q}^{-1}\underset{\sim}{C}^*\underset{\sim}{Q}'^{-1}(1 - \hat{\underset{\sim}{\beta}}')'} \tag{A-3}$$

Using the representation of $\underset{\sim}{Q}^{-1}$, we obtain

$$\frac{1}{\sigma}(1 - \hat{\underset{\sim}{\beta}}')\underset{\sim}{Q}^{-1} = (1\ 0) - \frac{1}{\mu}\hat{\underset{\sim}{e}}_{\beta}'(\underset{\sim}{f}^*\ \underset{\sim}{F}^*) \tag{A-4}$$

Substituting (A-4) into (A-3), the numerator divided by σ^2 is then

$$\underset{\sim}{x}'\underset{\sim}{x} - 2\hat{\underset{\sim}{e}}_{\beta}'\{\underset{\sim}{x}_1 + \frac{1}{\mu}\underset{\sim}{S}\underset{\sim}{x}\} + \hat{e}_{\beta}'\{\underset{\sim}{I} + \frac{1}{\mu}(\underset{\sim}{S}_1 + \underset{\sim}{S}_1') + \frac{1}{\mu^2}\underset{\sim}{S}\underset{\sim}{S}'\}\hat{\underset{\sim}{e}}_{\beta} \tag{A-5}$$

and the denominator devided by σ^2 is

$$c_{11}^* - \frac{2}{\mu}\hat{\underset{\sim}{e}}_{\beta}'(\underset{\sim}{f}^*\underset{\sim}{c}^*_{11} + F^*\underset{\sim}{c}_{21}^*) + \frac{1}{\mu^2}\hat{\underset{\sim}{e}}_{\beta}'(\underset{\sim}{f}^*\ \underset{\sim}{F}^*)\underset{\sim}{C}^*(\underset{\sim}{f}^*\ \underset{\sim}{F}^*)'\hat{\underset{\sim}{e}}_{\beta} \tag{A-6}$$

where $\underset{\sim}{S} = (\underset{\sim}{S}_1\ \underset{\sim}{S}_2)$, and $\underset{\sim}{S}_i = \underset{\sim}{f}^*\underset{\sim}{x}_i' + \underset{\sim}{F}^*\underset{\sim}{X}_i'$ for $i = 1,2$. The equations (A-5) and (A-6) are derived without using asymptotic assumptions, and they are just depend on a canonical representation of the model. Let us next write the expansion of the standardized estimator of $\underset{\sim}{\beta}$ as

$$\hat{\underset{\sim}{e}}_\beta = \underset{\sim}{e}_\beta^{(0)} + \frac{1}{\mu}\underset{\sim}{e}_\beta^{(1)} + \frac{1}{\mu^2}\underset{\sim}{e}_\beta^{(2)} + O_p(\mu^{-3}) \tag{A-7}$$

where each term on the RHS of (A-7) is obtained for the LIML and TSLS estimator in the large sample asymptotics (Fujikoshi et. al.) and for the LIML and fixed k-class estimators in the large μ^2 asymptotics (Kunitomo et. al.). The expression of $\underset{\sim}{e}_\beta^{(0)}$ is always $\underset{\sim}{x}_1$ for these estimators in both sequences. Substituting (A-7) into (A-5) the numerator becomes

$$\underset{\sim}{x}_2'\underset{\sim}{x}_2 - \frac{2}{\mu}\underset{\sim}{x}_1'\underset{\sim}{S}_2\underset{\sim}{x}_2 + \frac{1}{\mu^2}\{\underset{\sim}{e}_\beta^{(1)'}\underset{\sim}{e}_\beta^{(1)} + \underset{\sim}{x}_1'\underset{\sim}{S}\underset{\sim}{S}'\underset{\sim}{x}_1 + 2\underset{\sim}{e}_\beta^{(1)'}(\underset{\sim}{S}_1'\underset{\sim}{x}_1 - \underset{\sim}{S}_2\underset{\sim}{x}_2)\} \tag{A-8}$$

to terms of order μ^{-2} for all the estimators conceived in this paper for the both sequences. However, the expansions of the numerator does not hold the same expression as (A-8). For the large sample aymptotics (A-6) divided by (T - K) becomes

$$\begin{aligned}1 &+ \frac{1}{\mu}(\nu w_{11} - 2\underset{\sim}{x}_1'\underset{\sim}{f}^*) + \frac{1}{\mu^2}\{(\underset{\sim}{x}_1'\underset{\sim}{f}^*)^2 + \underset{\sim}{x}_1'\underset{\sim}{F}^*\underset{\sim}{F}^{*\prime}\underset{\sim}{x}_1 - 2\underset{\sim}{x}_1'(\underset{\sim}{f}^*\underset{\sim}{w}_{11} + \underset{\sim}{F}^*\underset{\sim}{w}_{21})\nu \\ &- 2\underset{\sim}{e}_\beta^{(1)'}\underset{\sim}{f}^* + \frac{\nu^2}{3}(w_{11}^2 - 2)\}\end{aligned} \tag{A-9}$$

to terms of order μ^{-2} for the TSLS and LIML estimators using (3.11) of Fujikoshi et. al. However (A-6) is

$$\begin{aligned}c_{11}^* &- \frac{2}{\mu}\underset{\sim}{x}_1'(\underset{\sim}{f}^*c_{11}^* + \underset{\sim}{F}^*\underset{\sim}{c}_{21}^*) + \frac{1}{\mu^2}\{c_{11}^*(\underset{\sim}{x}_1'\underset{\sim}{f}^*)^2 + 2(\underset{\sim}{x}_1'\underset{\sim}{f}^*)\underset{\sim}{c}_{12}^*\underset{\sim}{F}^{*\prime}\underset{\sim}{x}_1 \\ &+ \underset{\sim}{x}_1'\underset{\sim}{F}^*\underset{\sim}{C}_{22}^*\underset{\sim}{F}^{*\prime}\underset{\sim}{x}_1 - 2\underset{\sim}{e}_\beta^{(1)'}(\underset{\sim}{f}^*c_{11}^* + \underset{\sim}{F}^*\underset{\sim}{c}_{21}^*)\}\end{aligned} \tag{A-10}$$

to terms of μ^{-2} when μ^2 increases while the sample size stays fixed for the LIML and any fixed k-class estimators. The difference between (A-6) divided by T - K and (A-9) or between (A-6) and (A-10) are proved to be the order higher than μ^{-2} in each asymptotics. Multiplying Taylor's expansion of the inverse of (A-9) to (A-8) we obtain the large sample stochastic expansion of the variance ratio as

$$
\begin{aligned}
(T - K)\lambda = {} & x_2'x_2 - \frac{1}{\mu}\{\nu x_2'x_2 w_{11} + 2x_1'F^*X_2'x_2\} \\
& + \frac{1}{\mu^2}\{(e_\beta^{(1)} + S_1'x_1)'(e_\beta^{(1)} + S_1'x_1) - 2x_2'X_2F^{*\prime}(e_\beta^{(1)} + x_1 f^{*\prime}x_1) \\
& + (X_2F^{*\prime}x_1 + \nu w_{11}x_2)'(X_2F^*x_1 + \nu w_{11}x_2) \\
& + x_2'x_2\,[2x_1'F^*w_{21} - x_1'F^*F^{*\prime}x_1 - \frac{\nu^2}{3}(w_{11}^2 - 2)]\}
\end{aligned} \qquad \text{(A-11)}
$$

where the order of neglected terms is higher than μ^{-2}, and $\hat{e}_\beta^{(1)}$ can be one of $F^*X_2'x_2 - S_1'x_1$, $S_2x_2 - S_1'x_1$, $F^*X_2'x_2 - S_1'x_1 - hf^*$, $S_2x_2 - S_1'x_1 - (L + h)f^*$, or $F^*X_2'x_2 - S_1'x_1 + \frac{1}{L}x_2'x_2f^*$ according to the LIML, TSLS, ELI, ETS, or combined estimator, respectively. Similarly, the large μ^2 expansion of the variance ratio gives

$$
\begin{aligned}
s\lambda = {} & s\frac{x_2'x_2}{c_{11}^*} + \frac{2s}{\mu c_{11}^*}\{x_2'x_2x_1'F^*c_{21}^*/c_{11}^* - x_1'F^{*\prime}X_2'x_2\} \\
& + \frac{s}{\mu^2 c_{11}^*}\{(e_\beta^{(1)} + S_1'x_1)'(e_\beta^{(1)} + S_1'x_1) + x_1'F^*X_2'X_2F^{*\prime}x_1 + 2(x_1'f^*x_1 - e_\beta^{(1)})F^*X_2'x_2 \\
& + s\frac{x_2'x_2}{\mu^2 c_{11}^{*2}}\{2(x_1'f^*x_1 + e_\beta^{(1)})'F^*c_{21}^* + 4(x_1'F^*c_{21}^*)^2/c_{11}^* \\
& - x_1'F^*C_{22}^*F^{*\prime}x_1 - 4x_1'(f^*c_{11}^* + F^*c_{21}^*)x_1'F^*X_2'x_2\}
\end{aligned} \qquad \text{(A-12)}
$$

to terms of order μ^{-2} defining $s = (T - K)/L$. The terms neglected from (A-12) are proved to be higher order than μ^{-2}, and $e_{\beta}^{(1)}$ is one of

$F^*X_2'x_2 - S_1'x_1 - \frac{x_2'x_2}{c_{11}^*} F^*c_{21}^*$, $S_2x_2 - S_1'x_1 + (1 - k)(f^*c_{11}^* + F^*c_{21}^*)$, or

$F^*X_2'x_2 - S_1'x_1 + \frac{1}{L} x_2'x_2f^* - \frac{L-1}{L}\frac{x_2'x_2}{c_{11}^*} F^*c_{21}^*$ according to the LIML, fixed k-class, or combined estimator, respectively.

The rest of the computations are achieved by inverse Fourier type transformations of χ^2 and F random variables. We proceed the computation only for the LIML estimator in the two asymptotic theory. The same method of analysis holds for the variance ratios with other estimators. We denote the RHS of (A-11) and (A-12), for simplicity, as

$$\lambda^{(0)} + \frac{1}{\mu}\lambda^{(1)} + \frac{1}{\mu^2}\lambda^{(2)} \tag{A-13}$$

where $\lambda^{(i)}$ $i = 0,1,2$ correspond to the first, second, and third order terms of (A-11) or (A-12). We further define

$$v = x_2'x_2, \tag{A-14}$$

and

$$w = c_{11}^*. \tag{A-15}$$

In the large sample asymptotics $E(\lambda^{(1)}|v) = 0$,

$$E(\lambda^{(2)}|v) = L + 2(\nu^2 - 1)v, \tag{A-16}$$

and

$$E(\lambda^{(1)2}|v) = 4v + 2\nu^2v^2 \tag{A-17}$$

for λ with the LIML estimator. Then the expansion of the inverse transformation

$$\int_0^{\xi} \frac{1}{2\pi} \int_t \exp(-itx)\ E \exp\{it(T-K)\lambda_{LIML}\}dtdx$$

$$= \int_0^{\xi} \frac{1}{2\pi} \int_t \exp(-itx)\ E \exp(itv)\{1 + \frac{it}{\mu^2}E(\lambda^{(2)}|v) + \frac{1}{2}\frac{(it)^2}{\mu^2}E(\lambda^{(1)2}|v) + R\}dtdx \quad \text{(A-18)}$$

gives (3.4) applying the formulae in Lemma B-1 to each term on (A-18).

In the large μ^2 asymptotics $E(\lambda^{(1)}|v,w) = 0$,

$$E(\lambda^{(2)}|v,w) = -\frac{sv}{w}\ \{1 + \frac{v}{w} - \frac{L}{v} - \frac{4-(T-K)}{w}\} \quad \text{(A-19)}$$

and

$$E(\lambda^{(1)2}|v,w) = 4\ \{\frac{v^2}{w^3} + \frac{v}{w^2}\}. \quad \text{(A-20)}$$

Then the inverse transformation is

$$\int_0^{\xi} \frac{1}{2\pi} \int_t \exp(-itx)\ E \exp\{its\lambda_{LIML}\}dtdx$$

$$= \int_0^{\xi} \frac{1}{2\pi} \int_t \exp(-itx)\ E\exp(itsv/w)\{1 + \frac{it}{\mu^2}E(\lambda^{(2)}|v,w)$$

$$+ \frac{1}{2}\frac{(it)^2}{\mu^2}E(\lambda^{(1)2}|v,w) + R\}dtdx \quad \text{(A-21)}$$

which gives (3.2) applying the formulae in Lemma B-2 to each term on (A-21).

Appendix B

We give some useful formulae on the inverse transformation of characteristic functions related to the least variance ratios. The first lemma is on the approximate distributions about χ^2 distributions. The second lemma is on the approximate distributions about F distributions.

Lemma B.1: Let v be a χ^2 random variable with L degrees of freedom. Then

$$\int_{x=0}^{\xi} \frac{1}{2\pi} \int_t (-it)^p \exp(-itx) E[\exp(itv) v^j] dt dx$$

$$= 2^j \Gamma(L/2+j)/\Gamma(L/2) \cdot g_{L+2j}^{(p-1)}(\xi) \tag{B-1}$$

where $i = \sqrt{-1}$, j is any integer $(L + 2j > 0)$, and $g_{L+2j}^{(p-1)}(\xi)$ is the $(p - 1)$st order derivative of $g_{L+2j}(\xi)$ which is the χ^2 density function with L+2j degrees of freedom. In precise (B-1) with p = 1 is

$$\xi^j g_L(\xi) \tag{B-2}$$

and (B-1) with p = 2 is

$$\frac{1}{2}[(L + 2j - 2)\xi^{j-1} - \xi^j] g_L(\xi). \tag{B-3}$$

The equation (B-1) is obtained by inverse transformation of Laguerre polynomial multipliers.

Lemma B.2: Let v and w be independently distributed χ^2 random variables with L and q degrees of freedom. Then

$$\int_{x=0}^{\xi} \frac{1}{2\pi} \int_t (-it)^p \exp(-itx) E[\exp(itsv/w) v^j w^k] dt dx$$

$$= C a_{k,j}^p \, f_{L+2j,q+2k}^{(p-1)}(a_{k,j}\xi) \tag{B-4}$$

where $s = q/L$, $a_{k,j} = L(q + 2k)/[q(L + 2j)]$,
$c = 2^{j+k}\Gamma(L/2 + j)\Gamma(q/2 + k)/[\Gamma(L/2)\Gamma(q/2)]$, and $f_{L+2j,q+2k}^{(p-1)}(a_{k,j}\xi)$ is the $(p-1)$st derivative of the F density function with $L + 2j$ and $q + 2k$ degrees of freedom with respect to ξ. In particular (B-4) with $p = 1$ is

$$2^{j+k} \frac{\Gamma(a + j + k)}{\Gamma(a)} \frac{(\xi/s)^j}{(1 + \xi/s)^{j+k}} f_{L,q}(\xi), \tag{B-5}$$

and (B-4) with $p = 2$ is

$$2^{j+k-1} \frac{\Gamma(a + j + k)}{\Gamma(a)} \frac{(\xi/s)^j}{(1 + \xi/s)^{j+k}} \left[\frac{L + 2j - 2}{\xi} - \frac{L(L + q + 2j + 2k)}{q + L\xi}\right] f_{L,q}(\xi) \tag{B-6}$$

defining $a = (L + q)/2$.

<u>Proof</u>: sv/w is Fisher's F ratio with L and q degrees of freedom. Let its density function be $f_{L,q}(x)$. By the uniqueness of the inverse transformation,

$$\int_{x=0}^{\xi} \frac{1}{2\pi} \int_t^{\xi} \exp(-itx) \int_{w=0}^{\infty} \int_{v=0}^{\infty} \exp(its \frac{v}{w}) g_L(v) g_q(w) dv dt dx$$

$$= \int_0^{\xi} f_{L,q}(x) dx = F_{L,q}(\xi) \tag{B-7}$$

where the last equality is an identity defining the c.d.f. of sv/w.

By the property of χ^2 density function,

$$v^j w^k g_L(v) g_q(w) = c g_{L+2j}(v) g_{q+2k}(w).$$

Then

$$E[\exp(its \tfrac{v}{w}) v^j w^k]$$

$$= c \int_{w=0}^{\infty} \int_{v=0}^{\infty} \exp(its \tfrac{v}{w}) g_{L+2j}(v) g_{q+2k}(w) dv dw$$

$$= c \int_{w=0}^{\infty} \int_{v=0}^{\infty} \exp(it's' \tfrac{v}{w}) g_{L+2j}(v) g_{q+2k}(w) dv dw \qquad \text{(B-8)}$$

where $t' = \dfrac{t}{a_{k,j}}$, and $s' = \dfrac{q + 2k}{L + 2j}$.

Then (B-8) with (B-7) leads to

$$\int_{x=0}^{\xi} \frac{1}{2\pi} \int_t \exp(-itx) E[\exp(its \tfrac{v}{w}) v^j w^k] dt dx$$

$$= ca_{k,j} \int_x \frac{1}{2\pi} \int_{t'} \exp(-it'(a_{k,j}x)) \int_w \int_v \exp(it's' \tfrac{v}{w}) g_{L+2j}(v) g_{q+2k}(w) dv dw dt' dx$$

$$= ca_{k,j} \int_0^{\xi} f_{L+2j,q+2k}(a_{k,j}x) dx$$

$$= cF_{L+2j,q+2k}(a_{k,j}\xi) \qquad \text{(B-9)}$$

The pth orderivative of the first and third equations of (B-9) gives (B-5). When $p = 1$,

$$f_{L+2j,q+2k}(a_{k,j}\xi) = \frac{2^{j+k}}{ca_{k,j}} \frac{\Gamma(a + j + k)}{\Gamma(a)} \frac{(\xi/s)^j}{(1 + \xi/s)^{j+k}} f_{L,q}(\xi) \qquad \text{(B-10)}$$

which gives (B-5). Similarly when p = 2

$$f^{(1)}_{L+2j,q+2k}(a_{k,j}\xi) = \frac{1}{2a_{k,j}}\left[\frac{L + 2j - 2}{\xi} - \frac{L(L + q + 2k + 2j)}{q + L\xi}\right]f_{L+2j,q+2k}(\xi)$$

which gives (B-6) using (B-10).

Appendix C

This Appendix is about the validity of the asymptotic expansions formally derived in the Appendix A. The proof is essentially the same as that used by T. W. Anderson (1974) which involves expansions of the distribution function. We simply sketch the proof below confining ourselves to the asymptotic expansion of λ_{LIML} in the large sample sequence. The same results as the Theorems 1 and 2 were also obtained by the method explained below. However these computations include cumbersome statistical operations and are omitted from this article.

In the large sample asymptotics, we use the space J_T where each element of $\underset{\sim}{x}_1$, $\underset{\sim}{x}_2$, $\underset{\sim}{X}_1$, and $\underset{\sim}{X}_2$ is in the interval $(-2\sqrt{\log T}, 2\sqrt{\log T})$, and each element of $\underset{\sim}{W}$ (in particular W_{11} in this article) is in the interval $(-2\log T, 2\log T)$. Then $\Pr(J_T) = 1 - o(T^{-2})$ which is proved by Anderson (1974). In J_T, the numerator of λ_{LIML} and its demoninator divided by $(T - K)$ are expanded as (A-8) and (A-9) to terms of T^{-1}. The remainder terms are of order $T^{-3/2}$ in J_T. The long division of the inverse of (A-9) in J_T for sufficiently large T and multiplying it to (A-8) leads to (A-11). The remainder term of (A-11) is also of order $T^{-3/2}$ in J_T. Then the LHS of the inequality $(T - K)\lambda_{LIML} \leq \xi$ is replaced by the RHS of (A-11) with the remainder term of order $T^{-3/2}$. Define $\overline{P}_{X_2} = \underset{\sim}{I} - \underset{\sim}{X}_2(\underset{\sim}{X}_2'\underset{\sim}{X}_2)\underset{\sim}{X}_2'$ and $P_{X_2} = I - \overline{P}_{X_2}$. We note $\underset{\sim}{x}_2'\overline{P}_{X_2}\underset{\sim}{x}_2$, $\underset{\sim}{x}_2'P_{X_2}\underset{\sim}{x}_2$, and $\underset{\sim}{x}_2'\underset{\sim}{X}_2(\underset{\sim}{X}_2'\underset{\sim}{X}_2)^{-1/2}$ are independently distributed. The inequality which we want to compute probability is rewritten in the form as

$$\underset{\sim}{x}_2'\overline{P}_{X_2}\underset{\sim}{x}_2\{1 + \frac{1}{\mu} A + \frac{1}{\mu^2} B\} \leq (\xi - \underset{\sim}{x}_2'P_{X_2}\underset{\sim}{x}_2) + \frac{1}{\mu} C + \frac{1}{\mu^2} D + R \qquad \text{(C-1)}$$

by decomposing $x_2'x_2$ into the sum of $x_2'\overline{P}_{X_2}x_2$ and $x_2'P_{X_2}x_2$. The random variables in A, B, C, and D are independent of $x_2'\overline{P}_{X_2}x_2$. We devide both sides of (C-1) by $1 + A/\mu + B/\mu^2$ which is positive as T increases. By the long devision of $(1 + A/\mu + B/\mu^2)^{-1}$, (C-1) is finally written in the form as

$$x_2'\overline{P}_{X_2}x_2 \leqq (\xi - x_2'P_{X_2}x_2) + C'/\mu + D'/\mu^2 + R' \qquad \text{(C-2)}$$

where the remainder term is of order $T^{-3/2}$. Since $x_2'\overline{P}_{X_2}x_2$ is distributed as χ^2 with $(L - G_1)$ degrees of freedom (on the whole space),

$$\Pr\{(T - K)\lambda_{LIML} \leqq \xi\}$$

$$= E[G_{L-G_1}\{(\xi - x_2'P_{X_2}x_2) + C'/\mu + D'/\mu^2 + R'\}] \qquad \text{(C-3)}$$

where G_m is the χ^2 distribution function with m degrees of freedom. By Taylor's expansion of (C-3) in J_T, it is now

$$EG_{L-G_1}(\xi - x_2'P_{X_2}x_2) + E[g_{L-G_1}(\xi - x_2'P_{X_2}x_2)(C'/\mu + D'/\mu)]$$

$$+ E[g'_{L-G_1}(\xi - x_2'P_{X_2}x_2)(C'/\mu)^2] + R'' \qquad \text{(C-4)}$$

where $g_m(\ )$ is the χ^2 density function with m degrees of freedom and g_m' is its derivative. The new remainder term R" is of order $T^{-3/2}$. The expectation of (C-3) and (C-4) are over the whole space, and the difference between the integral over J_T and over the whole space is $o(T^{-2})$ because $G_m(\ )$ is bounded. The final result follows after some tedious computations of the first three terms in (C-4).

References

Anderson, T. W. (1974), "An Asymptotic Expansions of the Distributions of the Limited Information Maximum Likelihood Estimate of a Coefficient in a Simultaneous Equation System," Journal of the American Statistical Association, 9, 565-573.

_________ (1977), "Asymptotic Expansions of the Distributions of Estimates in Simultaneous Equations for Alternative Parameter Sequences," Econometrica, 45, 509-518.

_________ and Rubin, H. (1949), "Estimation of the Parameters of a Single Equation in a Complete System of Stochastic Equations," Annals of Mathematical Statistics, 21, 570-582.

_________ and Sawa, T. (1973), "Distributions of Estimates of Coefficients of a Single Equation in a Simultaneous System and Their Asymptotic Expansions," Econometrica, 41, 683-714.

_________, Morimune, K., and Sawa, T. (1978), "The Numerical Values of Some Key Parameters in Econometric Models," IMSSS Technical Report 270, Fourth Floor, Encina Hall, Stanford University, Stanford, California.

Basmann, R. L. (1960), "On Finite Sample Distributions of Generalized Classical Linear Identifiability Test Statistics," Journal of the American Statistical Association, 55, 650-659.

_________ (1965), "On the Application of the Identifiability Test Statistic in Predictive Testing of Explanatory Economic Models," The Econometric Annual of the Indian Economic Journal, 13, 387-423.

_________ (1966), "On the Application of the Identifiability Test Statistic in Predictive Testing of Explanatory Economic Models--Part II," The Econometric Annual of the Indian Economic Journal, 14, 233-252.

Fujikoshi, Yasunori (1977), "Asymptotic Expansions of the Distributions of the Latent Roots in MANOVA and the Canonical Correlations," Journal of Multivariate Analysis, 7, 386-396.

_________, Morimune, K., Kunitomo, N. and Taniguchi, M. (1978), "Asymptotic Expansions of the Distributions of the Estimates of Coefficients in a Simultaneous Equation System," Technical Report No. 287, Institute for Mathematical Studies in the Social Sciences, Fourth Floor, Encina Hall, Stanford University.

Fuller, Wayne A. (1977), "Some Properties of a Modification of the Limited Information Estimator," Econometrica, 45, 939-953.

Kadane, Joseph B. (1970), "Testing Over-identifying Restrictions When the Disturbances are Small," Journal of the American Statistical Association, 65, 182-185.

__________, and Anderson, T. W. (1977), "A Comment on the Test of Over-identifying Restrictions," Econometrica, 45, 1027-1031.

Kunitomo, N., Morimune, K., and Tsukuda, Y. (1979), "Asymptotic Expansions of the Distribution of Estimates of Coefficients in a Simultaneous Equation System When the Disturbances Are Small," in preparation.

McDonald, James B. (1972), "The Exact Finite Sample Distribution Function of the Limited Information Maximum Likelihood Identifiability Test Statistics," Econometrica, 40, 1109-1119.

Morimune, Kimio (1978), "Improving the Limited Information Maximum Likelihood Estimator When the Disturbances Are Small," Journal of the American Statistical Association, 73, 867-871.

__________, and Kunitomo, Naoto (1978), "Approximate Distributions of Identifiability Test Statistics for Single Structural Equations with Two Endogenous Variables," Discussion Paper No. 127, Kyoto Institute of Economic Research, Kyoto University.

Richardson, David H. (1968), "On the Distribution of the Identifiability Test Statistics," in J. P. Quirk and A. M. Zarley, eds., Papers in Quantitative Economics, Lawrence: The University Press of Kansas, pp. 203-216.

Rhodes, George F. (1978), "Exact Density and Approximate Critical Regions for Likelihood Ratio Identifiability Test Statistics," mimeographed.

Takeuchi, Kei and Morimune, Kimio (1979), "Asymptotic Completeness of the Extended maximum Likelihood Estimators in a Simultaneous Equation System," mimeographed.